Indian J. Phys. **76B (6)**, 747–751 (2002)

IJP B
— an international journal

Higher dimensional string cosmology in Lyra geometry

F Rahaman*[†], S Chakraborty[†], M Hossain, N Begum and J Bera
Khodar Bazar, Baruipur-743 302, 24 Parganas (South), West Bengal, India
[†]Department of Mathematics, Jadavpur University, Kolkata-700 032, India
E-mail : farook rahaman@yahoo.com

Received 16 May 2002, accepted 3 September 2002

Abstract : Some cosmological solutions for string model are derived in higher dimensional spherically symmetric space time based on Lyra's geometry. The physical behaviour of the models is also discussed

Keywords : String cosmology, Lyra geometry, higher dimension

PACS Nos. : 98.80.Cq, 04.20.Jb, 04.50.+h

1. Introduction

In last few years, there are attempts to unify gravity with other fundamental forces in nature. Latest studies of super string and super gravity theories and the unification of fundamental forces with gravity, reveal that the space time dimension should be different from four [1]. As a result, higher dimensional theory is receiving great attention both in Cosmology and in Particle Physics. It is argued that the extra dimensions are observable at the present time owing to their size being assumed to be of the order of the Planck length, but they may perhaps be relevant for the very early Universe [2]. The detection of time variation of fundamental constants may be a strong evidence for the existence of extra dimension [1,2].

The concept of string theory was developed to describe events at the early stages of the evolution of the Universe. It is believed that strings may be one of the sources of density perturbations that are required for formation of large scale structure in the Universe [3]. The existence of the large scale network of strings are well accustomed to the present day Universe.

Moreover, the vacuum strings can generate density fluctuations which explain the formation of the galaxy [4].These strings have stress energy and they couple to the gravitational field so that it may be interesting to study the gravitational effects which arise from strings. Cosmic strings as source of gravitational field in general relativity (GR) was discussed by many authors [5,6].

Since the discovery of general relativity theory by Einstein, there have been numerous modification of it. Lyra [7] proposed a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance to Weyl's geometry. Subsequent investigations were done by several authors [8] in scalar tensor theory and cosmology within the frame work of Lyra geometry.

But as far as our knowledge goes, there has not been any work in literature where Lyra's geometry has been considered for study of string cosmology in higher dimensional space time. Therefore, it is interesting to study string theory in higher dimension as both concepts are important at the early stages of the Universe.

In the present paper, we shall study string cosmology in higher dimensional space time based on Lyra's geometry in normal gauge *i.e.* displacement vector

$$\phi_1 = (\beta(t), 0, 0, 0, 0) \tag{1}$$

*Corresponding Author

© 2002 IACS

2. The basic equations

We consider a five dimensional space time with topology of 4-space S^1XS^3 as

$$ds^2 = -dt^2 + a^2 dr^2 + b^2 d\Omega_3^2, \tag{2}$$

where $a = a(t)$, $b = b(t)$ and

$$d\Omega_3^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2$$

is the metric on unit 3-sphere.

The coordinate r is periodic and varies in the range of $[0, 2\pi]$.

The field equations in normal gauge for Lyra's manifold as obtained by Sen [8] are

$$R_{ab} - \frac{1}{2} g_{ab} R + \frac{3}{2}\left(\phi_a\phi_b - \frac{1}{2} g_{ab}\phi_k\phi^k\right) = -\chi T_{ab}. \tag{3}$$

Here, ϕ_a is the displacement field vector defined in (1) and other symbols have their usual meanings as in Riemannian geometry. The energy momentum tensor for the string dust system is [6]

$$T_{ab} = \rho V_a V_b - \lambda x_a x_b, \tag{4}$$

where ρ is the rest energy density of the cloud of strings with particles attached to them (p-strings) and λ is the string's tension density. They are related by the relation

$$\rho = \rho_p + \lambda \tag{5}$$

with ρ_p being particle energy density. V_a is the five velocity for the cloud of particles and x_a is the direction of anisotropy *i.e.* the string's direction and they satisfy [6]

$$V_a V^a = -1 = -x_a x^a \text{ and } V_a x^a = 0$$

in (–, +, +, +, +) signature.

If we use comoving coordinate systems

i.e. $V^a = (1, 0, 0, 0, 0)$

and x^a parallel to $\frac{\partial}{\partial r}$ *i.e* $x_a = (0, a^{-1}, 0, 0, 0)$, then the nonvanishing components of field eq. (3) for the metric (2) are

$$\frac{3\dot{b}^2}{b^2} + \frac{3\dot{a}\dot{b}}{ab} + \frac{3}{b^2} - \frac{3}{4}\beta^2 = \rho. \tag{6}$$

$$\frac{3\ddot{b}}{b} + \frac{3\dot{b}^2}{b^2} + \frac{3}{b^2} + \frac{3}{4}\beta^2 = \lambda, \tag{7}$$

$$\frac{\ddot{a}}{a} + \frac{2\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} \cdot \frac{2\dot{a}\dot{b}}{ab} + \frac{1}{b^2} + \frac{3}{4}\beta^2 = 0. \tag{8}$$

The proper volume V^4, expansion scalar θ and shear scalar σ^2 are respectively given by

$$V^4 = ab^3, \tag{9}$$

$$\theta = \frac{\dot{a}}{a} + \frac{3\dot{b}}{b}, \tag{10}$$

$$\sigma^2 = \frac{\dot{a}^2}{a^2} + 3\frac{\dot{b}^2}{b^2} - \frac{1}{4}\theta^2. \tag{11}$$

The different equations of state for string model be [6]

(a) $\rho = \rho(\lambda)$ (barotropic equation of state),

(b) $\rho = \lambda$ (geometric string),

(c) $\rho = (1 + w)\lambda$ (Takabayasi string *i.e.* p-string).

In the following section, we shall determine the exact solutions of the field equations using above equations of state for string model in Lyra geometry.

3. Solutions

Case 1. Barotropic equation of state :

In this case, we take displacement vector to be constant *i.e.* β = constant.

To solve the field equations, one notes that there are three field equations connecting 4-unknowns. So one more relations connecting these variables is needed.

Here, we assume

$$a = \mu b^n \tag{12}$$

(μ, n are arbitrary constants) between the scale factors for unique solutions of the field equations.

Using this relation, we get from eq. (8)

$$\ddot{b} + A\frac{\dot{b}^2}{b} = -Bb - \frac{C}{b}, \tag{13}$$

where $A = \frac{n^2+n+1}{n+2}$, $B = \frac{3}{4}\cdot\frac{\beta^2}{n+2}$, $C = \frac{1}{n+2}$.

This equations has a first integral of the form

$$\dot{b}^2 = -\frac{Bb^2}{A+1} - \frac{C}{A} + Db^{-2A} \tag{14}$$

(where D is an integration constant).

This differential equation can be written in the integral form

$$\int \frac{db}{\left[Db^{-2A} - \frac{Bb^2}{A+1} - \frac{C}{A}\right]^{\frac{1}{2}}} = \pm(t - t_0) \tag{15}$$

(t_0 is another integration constant).

The above integral can be solved only when $A = 1$ *i.e.* $n = 1$.

Hence, we obtain,

$$b^2 = \frac{1}{B}\left[\sqrt{C^2 + 2BD}\sin\sqrt{2B}(t - t_0) - C\right]. \tag{16}$$

The other parameters have the following expressions

$$a = \frac{\mu}{\sqrt{B}}\left[\sqrt{C^2 + 2BD}\sin\sqrt{2B}(t - t_0) - C\right]^{\frac{1}{2}}, \tag{17}$$

$$V^4 = \mu\frac{1}{B^2}\left[\sqrt{C^2+2BD}\sin\sqrt{2B}(t-t_0)-C\right]^2, \quad (18)$$

$$\theta = \frac{2\sqrt{2BC^2+4BD^2}\cos\sqrt{2B}(t-t_0)}{\left[\sqrt{C^2+2BD}\sin\sqrt{2B}(t_0-t_0)-C\right]}, \quad (19)$$

$$\sigma^2 = 0, \quad (20)$$

$$\rho = \frac{3}{2}\frac{(2BC^2+4DB^2)\cos^2\sqrt{2B}(t-t_0)}{\left[\sqrt{C^2+2BD}\sin\sqrt{2B}(t-t_0)-C\right]^2}$$

$$+3B\left[\sqrt{C^2+2BD}\sin\sqrt{2B}(t-t_0)\right]^{-1}-\frac{3}{4}\beta^2, \quad (21)$$

$$\lambda = 2B\left[\sqrt{C^2+2BD}\sin\sqrt{2B}(t-t_0)-C\right]^{-1}, \quad (22)$$

$$\rho_p = \frac{3}{2}\frac{(2BC^2+4DB^2)\cos^2\sqrt{2B}(t-t_0)}{\left[\sqrt{C^2+2DB}\sin\sqrt{2B}(t_0-t_0)-C\right]^2}$$

$$+B\left[\sqrt{C^2+2BD}\sin\sqrt{2B}(t_0-t_0)-C\right]^{-1}-\frac{3}{4}\beta^2. \quad (22a)$$

Case II. Geometric string ($\rho = \lambda$) :

Here, we also assume the same relations between the metric coefficients *i.e.* $a = \mu b^n$, but the displacement vector is not constant. Using the above relation and after some calculatioris, we get

$$\frac{\ddot{b}}{b}+A\frac{\dot{b}^2}{b^2} = -\frac{2}{(2n+1)b^2}, \quad (23)$$

where $A = \frac{2n^2+5n+2}{2n+1}$.

Hence, b takes the following integral form

$$\int\frac{db}{\left[Db^{-2A}-\frac{2}{2n^2+5n+2}\right]^{\frac{1}{2}}} = \pm(t-t_0), \quad (24)$$

where D and t_0 are integration constants.

From the above integral equation, b can be obtained in closed form only for

$A = -1$, 1 and $\frac{1}{2}$ and we obtain

(i) $b = B\cosh\sqrt{D}(t-t_0)$, $B^2 = \dfrac{2}{2n^2+5n+2}$ (25)

(for $A = -1$) *i.e.* $n = -6$,

(ii) $b^2 = B^2 - \dfrac{2}{2n^2+5n+2}(t-t_0)^2$,

$$B^2 = \frac{D(2n^2+5n+2)}{2} \quad (26)$$

(for $A = 1$),

(iii) $\dfrac{-\sqrt{b}\sqrt{A_1^2-b}}{2}+\dfrac{A_1^2}{2}\left(\sin^{-1}\dfrac{b}{A_1}\right)$

$$= \frac{2\sqrt{2}}{\sqrt{2n^2+5n+2}}(t-t_0) \quad (27)$$

(for $A = \frac{1}{2}$),

$$A_1^2 = \frac{(2n^2 5n+2)D}{2}. \quad (28)$$

The other parameters are :

For $A = -1$:

$$a = \mu B^n\cosh^n\sqrt{D}(t-t_0), \quad (29)$$

$$\theta = (n+3)\sqrt{D}\tanh\sqrt{D}(t-t_0), \quad (30)$$

$$\sigma^2 = \frac{3}{4}(n-1)D\tanh^2\sqrt{D}(t-t_0), \quad (31)$$

$$V^4 = \mu B^{n+3}\left[\cosh\sqrt{D}(t-t_0)\right]^{n+3}, \quad (32)$$

$$\frac{3}{4}\beta^2 = 3nd\tanh^2\sqrt{D}(t-t_0)-3D, \quad (33)$$

$$\rho = \frac{3}{2}D+3D\tanh^2\sqrt{D}(t-t_0)$$

$$+\frac{3}{B^2}\mathrm{sec\,h}^2\sqrt{D}(t-t_0). \quad (34)$$

For $A = 1$, the solution shows a contracting model, and is not of much physical interest.

For $A = \frac{1}{2}$, we can not get explicit form of b in terms of t and consequently all physical parameters can not be determined in terms of t. Therefore, no physical conclusion can be drawn from the solution.

Case III. Takabayasi string (i.e. p-String) :

Here, the equation of state $\rho = \lambda(1 + w)$ where $w > 0$, a constant and it is small for string dominant era and large for particle dominant era.

Further, using the polynomial relation $a = \mu b^n$ between metric coefficients, from the field equations, we get

$$\frac{\ddot{b}}{b}+\frac{A\dot{b}^2}{b^2} = -\frac{B}{b^2}, \quad (35)$$

where $A = \dfrac{n^2(2+w)-n(2+w)+7n-2W+2nw+2}{2n+2wn+1-w}$,

$$B = \frac{2-2w}{2n+2wn+1-w}.$$

For $0 < w < 1$, the nature of the solutions is same as geometric string.

For $w = 1$, eq. (35) transforms to

$$\frac{\ddot{b}}{b} + \frac{A\dot{b}^2}{b^2} = 0,$$

where $A = \frac{3n+6}{4}$. (36)

Hence, we get

$$b = b_0 (t - t_0)^{\frac{4}{3n+10}}, \tag{37}$$

where t_0, b_{00} are integration constants and

$$b_0 = \left(b_{00}(A+1)\right)^{\frac{1}{A+1}}.$$

The other physical parameters are

$$a = \mu b_0^n (t - t_0)^{\frac{4n}{3n+10}}, \tag{38}$$

$$V^4 = \mu b_0^{n+3} (t - t_0)^{\frac{4(n+3)}{3n+10}}, \tag{39}$$

$$\theta = (n+3)\frac{4}{3n+10}\frac{1}{(t-t_0)}, \tag{40}$$

$$\sigma^2 = \frac{12(n-1)^2}{(3n+10)^2}\frac{1}{(t-t_0)^2}, \tag{41}$$

$$\frac{3}{4}\beta^2 = \frac{(36n^2+48n+120)}{(3n+10)^2(t-t_0)^2}, \tag{42}$$

$$\rho = \frac{3}{b_0^2}(t-t_0)^{-\frac{8}{3n+10}} + \frac{2(6-9n)}{(3n+10)^2}\frac{1}{(t-t_0)^2}. \tag{43}$$

For $w > 1$, we have the following integral form of b as

$$\int \frac{db}{\left[\frac{B}{A} + Db^{-2A}\right]^{\frac{1}{2}}} = \pm(t - t_0). \tag{44}$$

(D, t_0 are integration constants)

From the above integral equation, we obtain b for $A = -1$ and $+\frac{1}{2}$.

For $A = -1$, we have

$$b^2 = B(t-t_0)^2 - \frac{D}{B}. \tag{45}$$

$$a = \mu\left(B(t-t_0)^2 - \frac{D}{B}\right)^{\frac{n}{2}}, \tag{46}$$

$$\theta = \frac{(n+3)B(t-t_0)}{\left(B(t-t_0)^2 - \frac{D}{B}\right)^2}, \tag{47}$$

$$\sigma^2 = \frac{3(n-1)^2(t-t_0)^2}{4\left[B(t-t_0)^2 - \frac{D}{B}\right]^2}, \tag{48}$$

$$V^4 = \left[B(t-t_0)^2 - \frac{D}{B}\right]^{\frac{n+3}{2}}, \tag{49}$$

$$\beta^2 = \frac{2nB^2(t-t_0)^2}{\left[B(t-t_0)^2 - \frac{D}{B}\right]^2} - \frac{B}{2\left[B(t_0-t_0)^2 - \frac{D}{B}\right]}, \tag{50}$$

$$\rho = \frac{3B}{2\left[B(t-t_0)^2 - \frac{D}{B}\right]} + \frac{3B^2(t-t_0)^2}{\left[B(t-t_0)^2 - \frac{D}{B}\right]^2} + \frac{3}{\left[B(t-t_0)^2 - \frac{D}{B}\right]}. \tag{51}$$

For $A = +\frac{1}{2}$, we get

$$\sqrt{b}\sqrt{b + \frac{AD}{B}} - \frac{DA}{2B}\ln\left(\sqrt{b} + \sqrt{b + \frac{DA}{B}}\right) = \frac{1}{2}\sqrt{\frac{B}{A}}(t - t_0). \tag{52}$$

But in this case, b can not be expressed explicity as a function of t and consequently, all physical parameters can not be determined in terms of t.

4. Discussion

Case I :

In this model, we observe that

$$t = t_0 + \frac{1}{2B}\sin^{-1}\left(\frac{C}{\sqrt{C^2\_2BD}}\right)$$

is the initial epoch. The string model starts with an initial singularity $\sqrt{-g} \to 0$, while θ, ρ, λ diverge.

In fact, it is a point singularity as a, $b \to 0$ at this epoch.

Case II : (For $A = -1$, i.e. when $n = -6$) :

We note that the solutions in (25) and (29) describe a nonsingular space-time but shows a contracting model of the Universe and are not of much physically interest.

Case III : (For $w = 1$) :

The p-string will be interesting if $-3 < n < 0$. Then $t = t_0$ is the initial epoch of the Universe. At this instant, $b \to 0$, $a \to \infty$, $V^4 \to 0$, $\theta \to \infty$, $\rho \to 0$, $\beta^2 \to \infty$, and $\sigma^2 \to \infty$. So it is a line singularity. The contraction of extra dimension is possible in this case.

For $w > 1$ and $A = -1$, the nature of the solutions is interesting if $-3 < n < 0$.

In this case, we observe that at the initial epoch $(t - t_0)^2 = \frac{D}{B^2}$, the string model starts with an initial singularity

$\sqrt{-g} \to 0$ while $\theta, \sigma^2, \rho, \beta^2$ diverge. At this instant, $b \to 0$, $a \to \infty$, so it is a line singularity.

Thus, the universe starts with an infinite rate of expansion and measure of anisotropy.

The gauge function was large in begining but decreases with the evolution of the model.

Further, as t increases, the scale factor a gradually decreases while the other one namely b increases. This can be interpreted as follows : If we identify r - coordinate as the extra dimension, then it gradually decreases with the evolution of the Universe *i e* the extra dimension becomes unobservable small as the Universe evolves with time and we are left with usual 4-dimensional space and topology of 3-dimensional space is S^3

Therefore, we may conclude that the extra dimension is important at the very early stages of evolution and then it gradually becomes unobservably small as expected.

Acknowledgment

We are thankful to the members of Relativity and Cosmology Center, Jadavpur University. We are grateful to Dr. A A Sen for helpful discussion. FR is thankful to UGC for financial support and IUCAA for sending preprints and papers. Finally, we wish to thank the anonymous referee for pointing out an over sight which has led to stronger results than the one in the earlier version.

References

[1] T Appelquist, A Chodas and P G O Freund *Modern Kaluza Klein Theories* (Musachusetts : Addision Wesley) (1987)

[2] A Chodos and S Detweller *Phys. Rev* **D21** 2167 (1980)

[3] P S Letelier *Phys. Rev.* **D20** 1294 (1979); *Phys. Rev* **D28** 2414 (1983); *Stachel J Phys Rev.* **D21** 2171 (1980)

[4] T W B Kibble *J Phys.* **A9** 1387 (1976), Zeldovich Ya B *Mon. Not R Ast Soc.* **192** 663 (1980)

[5] A Vilenkin *Phys Rep* **121** 263 (1985)

[6] S Chakraborty *Indian J Pure Appl Phys.* **29** 31 (1991); *Astro Space Sci* **180** 293 (1991); S Chakraborty and A Chakraborty *J Math Phys.* **33** 2336 (1992); *Pramana* **40** 207 (1993), L K Patel and N Dadhich *Pramana* **47** 387 (1996), S Chakraborty and G C Nandy *Astro. Space Sci* **198** 299 (1992), *Pramana* **43** 503 (1993); S Chakraborty and A Gupta *Astro Space Sci* **239** 65 (1996); S Chakraborty and A Roy *Pramana* **51** 369 (1998)

[7] G Lyra *Maths. Z.* **59** 52 (1951)

[8] D K Sen *Phys. Z.* **149** 311 (1957), For *Brief notes on Lyra geometry* see also D K Sen and K A Dunn *J Math Phys* **12** 578 (1971); T Singh and G P Singh *Int J. Theo. Phys* **31** 1433 (1992); *J Math Phys* **32** 2456 (1991), *Nuovo Cim* **B106** 617 (1991); *Astro Space Sci.* **182** 201 (1991), G P Singh and K Desikan *Pramana* **49** 205 (1997), F Rahaman and I Bera *Int J. Mod. Phys.* **D10** 729 (2001), W D Halfold *Aust J Phys* **23** 8663 (1970); *J. Math Phys* **13** 1399 (1972), K Sheth and B B Waghmode *Gen. Rel. Grav* **14** 823 (1982); A Beesham *Astro. Space Sci* **127** (1987); *Aust J Phys* **41** 823 (1988), H H Soleng *Gen Rel. Grav* **19** 1213 (1987), T Karade and S Borikar *Gen. Rel Grav.* **9** 431 (1978); D R Reddy and P Innaiah *Astro. Space Sci* **123** 49 (1986), K Bharma *Aust J. Phys* **27** 541 (1974)